Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON INTEGRATING FACTORS

BY PAUL SAUREL

LIE has shown that if an ordinary differential equation of the first order in two variables remains invariant when subjected to a given infinitesimal transformation, it is possible to write at once an integrating factor of the equation; and Guldberg has called attention to the fact that this theorem can be extended to an integrable total differential equation in any number of variables. The following simple demonstration of Lie's theorem and of its generalization may be of interest.

Let us consider the ordinary differential equation of the first order in two variables:

$$Xdy - Ydx = 0. \tag{1}$$

The integral curves of this equation form a family which can be represented by the equation

$$f(x, y) = c, \tag{2}$$

in which c is an arbitrary parameter. At any point of one of these curves the direction cosines of the tangent are proportional to X and Y.

Instead of representing the integral curves by equation 2 we may represent them by the pair of equations

$$x = f_1(t, c), \qquad y = f_2(t, c). \tag{3}$$

To obtain any particular integral curve c must be kept constant while t varies.

Consider two consecutive integral curves corresponding to values c and $c + dc$ of the parameter. Let x, y be the coordinates of any point on the first curve and $x + dx$, $y + dy$ the coordinates of any neighboring point on the second curve. Through the point (x, y) draw the tangent to the first curve and on this tangent mark the point $(x + X, y + Y)$. The area of the parallelogram two of whose sides are the lines joining (x, y) with $(x + X, y + Y)$ and $(x + dx, y + dy)$ is

$$Xdy - Ydx. \tag{4}$$

Suppose now that in equations 3 we keep t constant and change c into

$c + dc$. We thus obtain a point on the second integral curve. If we denote the coordinates of this point by $x + \delta x$, $y + \delta y$ we have

$$\delta x = \frac{\partial f_1}{\partial c}\, dc, \quad \delta y = \frac{\partial f_2}{\partial c}\, dc. \tag{5}$$

But equations 3 enable us to express t and c in terms of x and y; we may therefore replace equations 5 by the following:

$$\delta x = \xi(x, y) dc, \qquad \delta y = \eta(x, y) dc. \tag{6}$$

The area of the parallelogram two of whose sides are the lines joining (x, y) with $(x + X, y + Y)$ and $(x + \delta x, y + \delta y)$ is

$$(X\eta - Y\xi) dc. \tag{7}$$

The two parallelograms which we have constructed are equal in area for they have the same base and a common altitude. From 4 and 7 we then obtain at once

$$\frac{Xdy - Ydx}{X\eta - Y\xi} = dc. \tag{8}$$

Since c is known to be a function of x and y it follows that the left side of equation 8 must be a perfect differential. The expression

$$\frac{1}{X\eta - Y\xi}$$

is therefore an integrating factor of equation 1.

We have thus established Lie's theorem:*

If the family of integral curves of the differential equation

$$Xdy - Ydx = 0$$

remains invariant when subjected to the infinitesimal transformation

$$\delta x = \xi(x, y) dc, \qquad \delta y = \eta(x, y) dc,$$

then

$$\frac{1}{X\eta - Y\xi}$$

is an integrating factor of the differential equation.

---

*Lie-Scheffers, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, p. 97; 1891.

J. M. Page, *Ordinary Differential Equations*, p. 75; 1897.

C. Jordan, *Cours d'analyse*, vol. 3, p. 21.

The demonstration just given cannot be extended immediately to total differential equations in three or more variables. But a slight modification in form will give us a demonstration which will apply to an integrable total differential equation in any number of variables.

Let us consider the integrable total differential equation in three variables:

$$X_1\,dx_1 + X_2\,dx_2 + X_3\,dx_3 = 0. \tag{9}$$

The integral surfaces of this equation form a family which can be represented by the equation

$$f(x_1, x_2, x_3) = c, \tag{10}$$

in which c is an arbitrary parameter. At any point of one of these surfaces the direction cosines of the normal are proportional to X_1, X_2 and X_3.

Instead of representing the integral surfaces by equation 10 we may represent them by the set of equations

$$x_1 = f_1(t_1, t_2, c), \quad x_2 = f_2(t_1, t_2, c), \quad x_3 = f_3(t_1, t_2, c). \tag{11}$$

To obtain any particular integral surface, c must be kept constant while t_1 and t_2 vary.

Consider two consecutive integral surfaces corresponding to values c and $c + dc$ of the parameter. Let x_1, x_2, x_3 be the coordinates of any point on the first surface and $x_1 + dx_1$, $x_2 + dx_2$, $x_3 + dx_3$ the coordinates of any neighboring point on the second surface. The length of the projection upon the normal to the first surface at the point (x_1, x_2, x_3) of the line joining (x_1, x_2, x_3) with $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ is equal to

$$\frac{X_1dx_1 + X_2dx_2 + X_3dx_3}{\sqrt{X_1^2 + X_2^2 + X_3^2}}. \tag{12}$$

Suppose now that in equations 11 we keep t_1 and t_2 constant and change c into $c + dc$. We thus obtain a point on the second integral surface. If we denote the coordinates of this point by $x_1 + \delta x_1$, $x_2 + \delta x_2$, $x_3 + \delta x_3$ we have

$$\delta x_1 = \frac{\partial f_1}{\partial c}dc, \quad \delta x_2 = \frac{\partial f_2}{\partial c}dc, \quad \delta x_3 = \frac{\partial f_3}{\partial c}dc. \tag{13}$$

But equations 11 enable us to express t_1, t_2 and c in terms of x_1, x_2 and x_3; we may therefore replace equations 13 by the equations

$$\delta x_1 = \xi_1(x_1, x_2, x_3)dc, \quad \delta x_2 = \xi_2(x_1, x_2, x_3)dc, \quad \delta x_3 = \xi_3(x_1, x_2, x_3)dc. \tag{14}$$

The length of the projection upon the normal to the first surface of the line joining (x_1, x_2, x_3) with $(x_1 + \delta x_1, \ x_2 + \delta x_2, \ x_3 + \delta x_3)$ is equal to

$$\frac{X_1\xi_1 + X_2\xi_2 + X_3\xi_3}{\sqrt{X_1^2 + X_2^2 + X_3^2}}\, dc. \tag{15}$$

The projections upon the normal of the two lines starting from (x_1, x_2, x_3) are equal in length, for each of these projections is equal to the portion of the normal which is intercepted between the two consecutive surfaces. From 12 and 15 we then obtain at once

$$\frac{X_1dx_1 + X_2dx_2 + X_3dx_3}{X_1\xi_1 + X_2\xi_2 + X_3\xi_3} = dc. \tag{16}$$

Since c is known to be a function of x_1, x_2, x_3 it follows that the left side of equation 16 must be a perfect differential. The expression

$$\frac{1}{X_1\xi_1 + X_2\xi_2 + X_3\xi_3}$$

is therefore an integrating factor of equation 9.

This demonstration has been stated in geometric terms but it is not hard to see that it can be divested of its geometric garb and stated in algebraic terms. To see this it will be sufficient to observe that the equality of expressions 12 and 15 can be established by showing that each is equal to

$$\frac{dc}{\sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + \left(\frac{\partial f}{\partial x_3}\right)^2}}$$

As soon as it is clear that the demonstration is independent of the geometric language in which it was clothed it becomes obvious that the demonstration remains valid whatever be the number of variables.

We have thus established Guldberg's generalization of Lie's theorem :*

If the family of integrals of the integrable total differential equation

$$\sum_{i=1}^{n} X_i dx_i = 0$$

---

* A. Guldberg, *Monatshefte für Mathematik und Physik*, vol. 7, p. 332; 1896.
J. M. Page, *Annals of Mathematics*, 1st Series, vol. 12, p. 182; 1899.

remains invariant when subjected to the infinitesimal transformation

$$\delta x_i = \xi_i(x_1, x_2, \cdots, x_n)dc, \qquad i = 1, 2, \cdots, n$$

then

$$\frac{1}{\sum_{i=1}^{n} X_i\xi_i}$$

is an integrating factor of the equation.

We shall conclude by calling attention to an interesting consequence of this theorem. Let us suppose that $X_1, X_2, \cdots, X_n$ are homogeneous functions of $x_1, x_2, \cdots, x_n$. It is well known that when $n = 2$ the infinitesimal transformation $\delta x_i = x_i dc$ leaves the family of integral curves invariant. It is very easy to show that in the general case the same transformation leaves the family of integrals invariant. We thus obtain Fais's theorem:*

If the coefficients $X_1, X_2, \cdots, X_n$ of an integrable total differential equation

$$\sum_{i=1}^{n} X_i dx_i = 0$$

are homogeneous functions of $x_1, x_2, \cdots, x_n$ then

$$\frac{1}{\sum_{i=1}^{n} X_i x_i}$$

is an integrating factor of the equation.

New York, February 4, 1905.

---

*A. R. Forsyth, *Theory of Differential Equations*, Part 1, p. 35.
J. M. Page, *Annals of Mathematics*, 1st Series, vol. 12, p. 182; 1899.